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Ciric Fixed Point Theorems in Metric Spaces with Binary Operation

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Abstract

This study extends recent developments in operational metric spaces, metric spaces endowed with an arbitrary binary operation, by introducing and proving new Ćirić fixed point theorems and Banach-like fixed point theorems within this framework. Operational metric spaces, recently formalized by Adewale et al. (2025), generalized classical metric spaces by allowing diverse binary operations, leading to broader applicability in fixed point theory. While prior work established several fixed point results in such spaces, it did not address the more general Ćirić-type contractions, which encompass a wide class of mappings beyond Banach contractions.
The research employs a rigorous analytical approach, proving multiple versions of Ćirić fixed point theorems under different binary operations; addition, maximum and minimum, alongside other Banach-type results. Each theorem is established by constructing appropriate iterative sequences, demonstrating their Cauchy property via the operational metric axioms, and applying completeness to guarantee convergence to a unique fixed point. Variants of contractive conditions, including max-based, min-based, and additive formulations, are systematically addressed, with detailed uniqueness proofs.
Key findings confirm that under suitable contractive conditions and binary operations, self-maps on complete operational metric spaces possess a unique fixed point. The results generalize several known theorems in classical metric spaces, b-metric spaces, and S-metric spaces, thereby broadening the scope of fixed point theory.
The study concludes that incorporating arbitrary binary operations into the metric framework not only preserves core fixed point properties but also enables more flexible and encompassing contractive mappings. These theorems unify and extend existing results, providing a foundation for further applications in nonlinear analysis, optimization, and other mathematical models where binary operations influence metric behaviour.

Keywords: Binary Operation, Fixed point, Ciric Type Contraction, Banach-type Contraction, Metric Space

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INTRODUCTION

Fixed point theory, a foundational branch of mathematical analysis, investigates the existence and properties of points that remain invariant under a given mapping. Since the pioneering work of Banach (1922), whose contraction mapping principle became a cornerstone of analysis, fixed point theory has evolved into an indispensable tool for solving problems in diverse fields such as differential and integral equations, optimization, game theory, and dynamical systems (Kirk, 2003; Rhoades, 2001). Over the decades, numerous generalizations of Banach's theorem have been established, each aiming to extend its applicability by relaxing conditions or redefining the underlying space (Karapınar, 2011; Jachymski, 1995).

One of the most influential generalizations was introduced by Ćirić (1974), who formulated a broader class of contractive mappings, now known as Ćirić-type contractions. These mappings extend Banach contractions by incorporating additional control parameters and permitting weaker contractive conditions, thus accommodating a larger set of operators (Berinde, 2007; Kumam et al., 2012). Subsequent research has produced numerous refinements, exploring Ćirić-type results in various generalised metric frameworks, such as b-metric spaces (Czerwik, 1993), G-metric spaces (Mustafa & Sims, 2006), S-metric spaces (Sedghi et al., 2012), and cone metric spaces (Huang & Zhang, 2007). A recent innovation in this domain is the concept of operational metric spaces, introduced by Adewale et al. (2025). These spaces retain the fundamental axioms of metric spaces but are augmented with an arbitrary binary operation on the set. This modification enables the metric to interact with the binary operation in defining convergence, completeness, and contractive conditions. The binary operation, whether addition, maximum, minimum or another admissible operation, provides a richer structural framework, unifying and extending several known metric-type spaces (Adewale et al., 2025).

Although Banach-type fixed point results have been established in operational metric spaces (Adewale et al., 2025), there remains a significant gap in the literature: the absence of Ćirić-type fixed point theorems in this setting. Given the generality of Ćirić contractions and their ability to subsume various existing contractive mappings, extending them to operational metric spaces is both a natural and necessary progression. Such an extension not only broadens the theoretical foundation but also enhances the potential for applications in areas where binary operations are intrinsic, such as ordered algebraic structures, computational models and certain classes of functional equations (Choudhury & Kundu, 2016; Karapınar & Piri, 2014).

The purpose of this study is to fill this gap by developing and proving new Ćirić fixed point theorems and Banach-like results in operational metric spaces under different binary operations, including addition, maximum and minimum. By doing so, we generalised existing results from classical and generalised metric spaces, offering a unified framework that accommodates a wider class of mappings. This contribution not only enriches the theoretical landscape but also provides a versatile toolset for future studies in nonlinear analysis and applied mathematics. Specifically, in Biomathematics. (Loyinmi & Oredein,2011; Loyinmi, 2024a; Loyinmi, 2024b; Loyinmi et al., 2025; Loyinmi et al., 2018; Loyinmi & Idowu, 2023;).

PRELIMINARIES

We introduced the following:

Definition 1.1: Let K be a non-empty set, $\ominus$, a binary operation with e as its identity element and $\theta : K^2 \rightarrow \mathbb{R}^+$. θ is called an operational metric if the following axioms are satisfied:

$\theta_1 : \theta(m,n) \geq e$;

$\theta_2 : \theta(m,n) = e \; if \; and \; only \; if \; m = n$;

$\theta_3 : \theta(m,n) = \theta(m,n)$;

$\theta_4 : \theta(m,n) \leq \theta(m,r) \otimes b(r,n) \; for \; all \; m,n,r \in K$.

K together with θ is called an operational metric space. Denoted by $(K, \theta, \ominus)$

Remark 1.2:

i. If the binary operation $\ominus$ is defined by p $\ominus$ q = p + q, the Definition 1.1 reduces to metric space introduced by Fretchet (1906).
ii. If the binary operation $\ominus$ is defined by p $\ominus$ q = p × q, the Definition 1.1 reduces to b- metric space introduced by Bakhtin (1989).

Example 1.3: Let $K = \{p \in N : 3 \leq p \leq 9\}$ and the binary operation $\ominus$ be defined by

$$p \ominus q = p + q - 3.$$

If $\theta(p,q) = |p-q| + 3$, then θ is an operational metric and $(K, \theta, \ominus)$ is an operational metric space.

Verification:

i. By definition

$$|p - q| = \begin{cases} p - q, if \; p - q \geq 0 \\ q - p, if \; p - q < 0 \end{cases}$$

So, $|p - q| \geq 0$.

Since, $|p - q| \geq 0$,

$$|p-q| + 3 \geq 3 \; for \; all \; p \in K.$$

Hence, $\theta(p,q) = |p - q| + 3$

$$\geq e = 3.$$

If $p \ominus e = p$, then

$p + e - 3 = p \Longrightarrow e = 3$.

ii. $\theta(p,q) = e$

$$\Longrightarrow |p-q| + 3 = e \Longrightarrow$$

$$|p-q| = 0 \Longrightarrow p = q.$$

Conversely, If $p = q$,

Then $p - q = 0$

$\Longrightarrow |p-q| = 0$

$\Longrightarrow |p-q| + 3 = e$

$\Longrightarrow \theta(p,q) = e$

iii. $\theta(p,q) = |-(p - q)| + 3 = |-p+q| + 3 = |q-p| + 3 = \theta(q,p)$.

iv.
$$\theta(p,q) = |p-q| + 3 \quad (1)$$
$$= |p-u+u-q| + 3 \quad (2)$$
$$\leq |x-a| + |a-y| + 3 \quad (3)$$
$$< |x-a| + 3 + |a-y| + 3 \quad (4)$$
$$= b(x,a) + b(a,y). \quad (5)$$

Example 1.4: Let $K = R$ and the binary operation $\ominus$ be defined by $p \ominus q = p + q$.

If $\theta(p, q) = |p - q|$, then θ is an operational metric and $(K, \theta, \ominus)$ is an operational metric space.

Definition 1.5: Let $(K, \theta, \ominus)$ be an operational metric space. An open sphere centered at p with radius u in K is defined by

$$S_u(p) = \{v : \theta(p, v) < u$$

Definition 1.6: Let $(K, \theta, \ominus)$ be an operational metric space. A closed sphere centered at p with radius u in K is defined by

$$S_u[p] = \{v : \theta(p, v) \leq u\}$$

Definition 1.7: Let $(K, \theta, \ominus)$ be an operational metric space. A sphere centered at p with radius u in K is defined by

$$S(u, p) = \{v : \theta(p, v) = u\}$$

Definition 1.8: Let $(K, \theta, \ominus)$ be an operational metric space and $\{p_n\}$, a sequence in K. A sequence, $\{p_n\}$ converge to w if for

$$n \in \mathbb{N}, \theta(p_n, w) \rightarrow e \, as \, n \rightarrow \infty.$$

Definition 1.9: Let $(K, \theta, \ominus)$ be an operational metric space and $\{p_n\}$, a sequence in K. A sequence, $\{p_n\}$ in K is said to be a Cauchy sequence if for $n, m \in \mathbb{N}$ with

$$n > m, \theta(p_n, q_m) \rightarrow e \text{ as } n, m \rightarrow \infty.$$

Definition 1.10: Let $(K_1, \theta_1, \ominus)$ and $(K_2, \theta_2, \ominus)$ be two operational metric spaces. $A\ g : K_1 \rightarrow K_2$ is said to be continuous at a point $w \in K_1$ if for all ϵ > e the exists δ > e such that

$$\theta_1(s, w) < \delta \Longrightarrow \theta_2(g(s), g(w)) < \epsilon.$$

The function g is continuous on K_1 if it is continuous at every point $w \in K_1$.

RESEARCH METHODOLOGY

The research adopts a theoretical and constructive approach. First, relevant definitions and lemmas are established to lay the groundwork for the main results. Then, generalised Ciric fixed point theorems are proved within the context of metric spaces equipped with a binary operation. The proofs utilized iterative techniques, completeness arguments, and the properties of the binary operation to ensure convergence to a fixed point. Examples are constructed to demonstrate the applicability of the theorems in diverse scenarios. The results were compared to existing literature to highlight the generalization achieved

MAIN RESULTS

In 2025, Adewale et.al. introduced metric space with binary operation. In this new space, they introduced some fixed point theorems which did not include Ciric fixed point theorem and Ciric fixed point theorem is a generalisation of those contractions in (Adewale et al., 2025; Ayodele et al., 2025; Adewale et al., 2024; Adewale & Akinremi, 2013; Agarwal et al., 2001; Agarwal et al., 2009; Zhou et al ., 2017; Olaleru & Akewe, 2019; Chidume, 2004; Olaleru, 2007) for more understanding.

In this paper, we introduced the Ciric fixed point theorems.

Theorem 4.1: Let (K, θ, $\ominus$) be a complete operational metric space in which the binary operation is defined by p $\ominus$ q = p + q. Suppose T : K → K is a self-map. Assume that there exists a constant c with $c \in [0, 0.5)$ such that, for all p, q ∈ K, the following inequality holds:

θ (Tp, Tq) ≤ c · max{ θ (p, q), θ (p, Tq), θ (q, Tp), θ (p, Tp), θ (q, Tq) }. (6)
Then T has a unique fixed point in K.

Proof: Considering (1) with an arbitrary point $p_0 \in K$ and define a sequence p_n by

$$p_n = T^n p_0$$

$$\theta(p_n, p_{n+1}) = \theta(T_{p_{n-1}}, T_{p_n}) \quad (7)$$

$$\leq c\, max\{\theta(p_{n-1}, Tp_n), \theta(p_n, Tp_{n-1}), \theta(p_{n-1}, p_n), \theta(p_{n-1}, Tp_{n-1}), \theta(p_n, Tp_n)\} \quad (8)$$

$$= c\, max\{\theta(p_{n-1}, p_{n+1}), \theta(p_n, p_n), \theta(p_{n-1}, p_n), \theta(p_{n-1}, p_n), \theta(p_n, p_{n+1})\} \quad (9)$$

$$= c\, max\{\theta(p_{n-1}, p_{n+1}), \theta(p_{n-1}, p_n), \theta(p_n, p_{n+1})\} \quad (10)$$

$$= c\, \theta(p_{n-1}, p_{n+1}) \quad (11)$$

$$\leq c[\theta(p_{n-1}, p_n) + \theta(p_n, p_{n+1})] (12)$$

(2) implies

$$\theta(p_n, p_{n+1}) \leq \frac{c}{1-c}\theta(p_{n-1}, p_n). \quad (13)$$

If $l = \frac{k}{1-k}$, then

$$\theta(p_n, p_{n+1}) \leq l\left(\theta(p_{n-1}, p_n)\right). \quad (14)$$

Suppose T satisfies condition (4), then

$$\theta(p_n, p_{n+1}) \leq l\left(\theta(p_{n-1}, p_n)\right). \quad (15)$$

$$\leq l^2\, (b(x_{n-1}, x_n)) (16)$$

Using this repeatedly, we obtain

$$\theta(p_n, p_{n+1}) \leq l^n\left(\theta(p_0, p_1)\right). (17)$$

By using (θ_4) of Definition 1.1 with $n > m$, we have

$$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) \ominus \theta(p_{n-1}, p_m) \quad (18)$$

$$= \theta(p_n, p_{n-1})\theta(p_{n-1}, p_m) (19)$$

$$\theta(p_n, p_{n-1})\theta(p_{n-1}, p_{n-2}) + \ldots + \theta(p_{m+1}, p_m) \quad (20)$$

With (7) and (10), we obtain

$$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) + \theta(p_{n-1}, p_{n-2}) + \ldots + \theta(p_{m+1}, p_m) \quad (21)$$

$$\leq l^{n-1}\theta(p_0, p_1) + l^{n-2}\theta(p_0, p_1) + \ldots + l^m\theta(p_0, p_1) \quad (22)$$

$$\leq [l^{n-1} + l^{n-2} + \ldots + l^m]\theta(p_0, p_1) \quad (23)$$

$$\leq l^n[c^{-1} + l^{-2} + \ldots + l^{m-n}]\theta(p_0, p_1) \quad (24)$$

$$\leq \frac{l^n}{l-1}\theta(p_0, p_1) \quad (25)$$

Taking the limit of $\theta(p_n, p_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e. \quad (26)$$

So, $\{p_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is Convergent to s.

Suppose $Ts \neq s$

$$\theta(p_n, Ts) \leq c\, max\{\theta(p_{n-1}, s), \theta(p_{n-1}, Ts), \theta(s, Tp_{n-1}), \theta(p_{n-1}, Tp_{n-1}), \theta(s, Ts)\}$$

Taking the limit as $n \to \infty$ and using the fact that the function is continuous in its variables, we get

$$\theta(s, Ts) \leq c\left(\theta(s, Ts)\right) \quad (27)$$

Hence,

$$\theta(s, Ts) \leq e \quad (28)$$

This is a contraction. So, $Ts = s$

To show the uniqueness, suppose $r \neq s$ is such that $Tr = r$ and $Ts = s$,

Then

$$\theta(Ts, Tr) \leq c(\theta(s, r)). \quad (29)$$

Since $Ts = s$ and $Tr = r$, we have

$$\theta(s, r) \leq e. \quad (30)$$

which implies that $r = s$.

Theorem 4.2: Let $(K, \theta, \ominus)$ be a complete operational metric space with an operation defined by $p \ominus q = max\{p, q\}$. Suppose $T : K \to K$ is a self map and there exists a real number c, satisfying $0 \leq c < 0.5$ for each $p, q \in K$ with

$$\theta(Tp, Tq) \leq c\, max\{\theta(p, q), \theta(p, Tq), \theta(q, Tp), \theta(p, Tp), \theta(q, Tq)\}. \quad (31)$$

Then T has a unique fixed point.

Proof: Considering (21) with an arbitrary point $p_0 \in K$ and define a sequence p_n by $p_n = T^n p_0$,

$$\theta(p_n, p_{n+1}) = \theta(Tp_{n-1}, Tp_n) \quad (32)$$

$$\leq kmax\{\theta(Tp_{n-1}, Tp_n), \theta(p_n, Tp_{n-1}), \theta(p_{n-1}, p_n), \theta(p_{n-1}, Tp_{n-1}), \theta(p_n, Tp_n)\} \quad (33)$$

$$= c\, max\{\theta(p_{n-1}, p_{n+1}), \theta(p_n, p_n), \theta(p_{n-1}, p_n), \theta(p_{n-1}, p_n), \theta(p_n, p_{n+1})\} \quad (34)$$

$$= \text{c max}\{\theta(p_{n-1}, p_{n+1}), \theta(p_{n-1}, p_n), \theta(p_n, p_{n+1})\} \quad (35)$$

$$= \text{c}\, \theta(p_{n-1}, p_{n+1}) \quad (36)$$

$$\leq \text{c}[\theta(x_{n-1}, x_n) + \theta(x_n, x_{n+1})]. \quad (37)$$

(22) Implies

$$\theta(p_n, p_{n+1}) \leq \frac{c}{1-c}\theta(p_{n-1}, p_n) \quad (38)$$

If $l = \frac{c}{1-c}$, then

$$\theta(p_n, p_{n+1}) \leq l\theta(p_{n-1}, p_n) \quad (39)$$

Suppose T satisfies condition (24), then

$$\theta(p_n, p_{n+1}) \leq l(\theta(p_{n-1}, p_n)) \quad (40)$$

$$\leq l^2(\theta(p_{n-2}, p_{n-1})) \quad (41)$$

Using this repeatedly, we obtain

$$\theta(p_n, p_{n+1}) \leq l^n(\theta(p_0, p_1)). \quad (42)$$

By using (θ_4) of Definition 1.1 with $n > m$, we have

$$\theta(p_n, p_m) \leq (p_n, p_{n-1}) \ominus (p_{n-1}, p_m) \quad (43)$$

$$= max\{b(p_n, p_{n-1}), \theta(p_{n-1}, p_m)\} \quad (44)$$

$$= max\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{m+1}, p_m)\} \quad (45)$$

With (27) and (30), we obtain

$$\theta(p_n, p_m) \leq max\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{m+1}, p_m)\} \quad (46)$$

$$\leq max\begin{Bmatrix} l^{n-1}\theta(p_0, p_1), l^{n-2}\theta(x_0, x_1), \\ \ldots, l^m b(x_0, x_1) \end{Bmatrix} \quad (47)$$

$$\leq \{l^{n-1}, l^{n-2}, \ldots, l^m\}\theta(p_0, p_1) \quad (48)$$

Taking the limit of $\theta(p_n, p_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e. \quad (49)$$

So, $\{p_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is convergent to s.

Suppose $Ts \neq s$

$$\theta(p_n, Ts) \leq c\,max\{\theta(p_{n-1}, s), \theta(p_{n-1}, Ts), \theta(s, Tp_{n-1}), \theta(p_{n-1}, Tp_{n-1}), \theta(s, Ts)\}.$$

Taking the limit as $n \to \infty$ and using the fact that the function is continuous in its variables,

we get

$$\theta(s, Ts) \leq c(\theta(s, Ts)) \quad (50)$$

Hence,

$$\theta(s, Ts) \leq e. \quad (51)$$

This is a contradiction. So, $Ts = s$.

To show the uniqueness, suppose $w \neq s$ is such that $Tw = w$ and $Ts = s$,

Then,

$$\theta(Ts, Tw) \leq c(\theta(s, w)). \quad (52)$$

Since $Ts = s$ and $Tw = w$, we have

$$\theta(s, w) \leq e. \quad (53)$$

which implies that $w = s$.

Theorem 4.3: Let $(K, \theta, \ominus)$ be a complete operational metric space with an operation defined by $p \ominus q = min\{p, q\}$. Suppose $T : K \to K$ is a self map and there exists a real number c, satisfying $0 \leq c < 0.5$ for each $p, q \in K$ with

$$\theta(Tp, Tq) \leq c\,max\{\theta(p, q), \theta(p, Tq), \theta(q, Tp), \theta(p, Tp), \theta(q, Tq)\} \quad (54)$$

Then T has a unique fixed point.

Proof: Considering (39) with an arbitrary point $p_0 \in K$ and define a sequence p_n

By $p_n = T^n p_0$,

$$\theta(p_n, p_{n+1}) = \theta(Tp_{n-1}, Tp_n) \quad (55)$$

$$\leq c\,max\{\theta(p_{n-1}, Tp_n), \theta(p_n, Tp_{n-1}), \theta(p_{n-1}, p_n), \theta(p_{n-1}, Tp_{n-1}), \theta(p_n, Tp_n)\} \quad (56)$$

$$= c\,max\{\theta(p_{n-1}, p_{n+1}), \theta(p_n, p_n), \theta(p_{n-1}, p_n), \theta(p_{n-1}, p_n), \theta(p, p_{n+1})\} \quad (57)$$

$$= c\,max\{\theta(p_{n-1}, p_{n+1}), \theta(p_{n-1}, p_n), \theta(p_n, p_{n+1})\} \quad (58)$$

$$= c\,\theta(p_{n-1}, p_{n+1}) \quad (59)$$

$$\leq c[\theta(p_{n-1}, p_n) + \theta(p_n, p_{n+1})] \quad (60)$$

(40) Implies

$$\theta(p_n, p_{n+1}) \leq \frac{c}{1-c}\,\theta(p_{n-1}, p_n). \quad (61)$$

If $l = \frac{c}{1-c}$, then

$$\theta(p_n, p_{n+1}) \leq l\theta(p_{n-1}, p_n). \quad (62)$$

Suppose T satisfies condition (42), then

$$\theta(p_n, p_{n+1}) \leq l(\theta(p_{n-1}, p_n)) \leq l^2(\theta(p_{n-2}, p_{n-1})) \quad (63)$$

Using this repeatedly, we obtain

$$\theta(p_n, p_{n+1}) \leq l^n(\theta(p_0, p_1)). \quad (64)$$

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By using (θ_4) of Definition 1.1 with $n > m$, we have

$$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) \ominus \theta(p_{n-1}, p_m) \quad (65)$$

$$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_m)\} \quad (66)$$

$$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{n+1}, p_m)\} \quad (67)$$

With (45) and (48), we obtain

$$\theta(p_n, p_m) \leq min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{n+1}, p_m)\} \quad (68)$$

$$\leq min\{l^{n-1}\theta(p_0, p_1), l^{n-2}\theta(p_0, p_1), \ldots, l^m\theta(p_0, p_1)\} \quad (69)$$

$$\leq min\{l^{n-1}, l^{n-2}, \ldots, l^m\}\theta(p_0, p_1) \quad (70)$$

Taking the limit of $\theta(p_n, p_m)$) as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e. \quad (71)$$

So, $\{p_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is convergent to s.

Suppose $Ts \neq s$

$$\theta(p_n, Ts) \leq cmax\{\theta(p_{n-1}, s), \theta(p_{n-1}, Ts), \theta(s, Tp_{n-1}), \theta(p_{n-1}, Tp_{n-1}), \theta(s, Ts)\}. \quad (72)$$

Taking the limit as $n \to \infty$ and using the fact that the function is continuous

in its variables, we get

$$\theta(s, Ts) \leq c(\theta(s, Ts)). \quad (73)$$

Hence,

$$\theta(s, Ts) \leq e. \quad (74)$$

This is a contradiction. So, $Ts = s$.

To show the uniqueness, suppose $w \neq s$ is such that $Tw = w$ and $Ts = s$,

then

$$\theta(Ts, Tw) \leq c(\theta(s, w)). \quad (75)$$

Since $Ts = s$ and $Tw = w$, we have

$$\theta(s, w) \leq e. \quad (76)$$

This implies that $w = s$.

Theorem 4.4: Let $(K, \theta, \ominus)$ be a complete operational metric space with an operation defined by $p \ominus q = min\{p, q\}$. Suppose is a self map and there exists a real number c, satisfying $0 \leq c < 1$ for each $p, q \in K$ with

$$\theta(Tp, Tq) \leq c(\theta(p, q)). \quad (77)$$

Then T has a unique fixed point.

Proof: Considering (57) with an arbitrary point $p_0 \in K$ and define a sequence p_n

by $p_n = T^n p_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k(b(x_{n-1}, x_n)) \quad (78)$$

Suppose T satisfies condition (58),

then

$$\theta(p_n, p_{n+1}) = \theta(Tp_{n-1}, Tp_n) \quad (79)$$
$$\leq c(\theta(p_{n-1}, p_n)) \quad (80)$$

$$\leq c^2(\theta(p_{n-2}, p_{n-1})) \quad (81)$$

Using this repeatedly, we obtain

$$\theta(p_n, p_{n+1}) \leq c^n(\theta(p_0, p_1)). \quad (82)$$

By using (θ_4) of Definition 1.1 with $n > m$, we have

$$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) \ominus \theta(p_{n-1}, xp_m) \quad (83)$$

$$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_m)\} \quad (84)$$

$$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}),. \quad .., b(p_{m+1}, p_m)\} \quad (85)$$

With (62) and (65), we obtain

$$\theta(p_n, p_m) \leq min\left\{\begin{matrix}\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \\ \dots, \theta(p_{m+1}, p_m)\end{matrix}\right\} \quad (86)$$

$$\leq min\{c^{n-1}\theta(p_0, p_1), c^{n-2}\theta(p_0, p_1),. \quad .., c^m\theta(p_0, p_1)\} \quad (87)$$

$$\leq min\{c^{n-1}, c^{n-2}, \dots, c^m\}\theta(p_0, px_1) \quad (88)$$

Taking the limit of $\theta(p_n, p_m)\ as\ n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e. \quad (89)$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is convergent to s.

Suppose $Ts \neq s$

$$\theta(p_n, Ts) \leq\leq k(b(x_{n-1}, x_n)(80) \quad \leq k^2(b \quad c(\theta(p_{n-1}, s)). \quad (90)$$

Taking the limit as $n \to \infty$ and using the fact that the function is continuous

in its variables, we get

$$\theta(s, Ts) \leq c(\theta(s, Ts)). \quad (91)$$

Hence,

$$\theta(s, Ts) \leq e \quad (92)$$

This is a contradiction. So, $Ts = s$.

To show the uniqueness, suppose $w \neq s$ is such that $Tw = w$ and $Ts = s$,

Then,

$$\theta(Ts, Tw) \leq c(\theta(w, s)). \quad (93)$$

Since $Ts = s$ and $Tw = w$, we have

$$\theta(s, w) \leq e. \quad (94)$$

This implies that $w = s$.

Theorem 4.5: Let $(K, \theta, \ominus)$ be a complete operational metric space with an operation defined by $p \ominus q = min\{p, q\}$. Suppose $T : K \to K$ is a self map and there exists a real number c, satisfying $0 \leq c < 0.5$ for each $p, q \in K$ with

$$\theta(Tp, Tq) \leq c[\theta(p, Tp) + \theta(q, Tq)]. (95)$$

Then T has a unique fixed point.

Proof: Considering (75) with an arbitrary point $p_0 \in K$ and define a sequence $\{p_n\}$

By $p_n = T^n p_0$,

$$\theta(p_n, p_{n+1}) = \theta(Tp_{n-1}, Tp_n) \leq c[\theta(p_{n-1}, p_n) + \theta(p_{n+1}, p_n)]. \quad (96)$$

(76) implies

$\theta(p_n, p_{n+1}) \leq \frac{c}{1-c} \theta(p_{n-1}, p_n)$. (97)

If $l = \frac{c}{1-c}$, then

$\theta(p_n, p_{n+1}) \leq l\theta(p_{n-1}, p_n)$. (98)

Suppose T satisfies condition (78), then

$\theta(p_n, p_{n+1}) \leq l(\theta(p_{n-1}, p_n))$(99)
$\leq l^2(\theta(p_{n-2}, p_{n-1}))$ (100)

Using this repeatedly, we obtain

$\theta(p_n, p_{n+1}) \leq l^n(\theta(p_0, p_1))$. (101)

By using (θ_4) of Definition 1.1 with $n > m$, we have

$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) \ominus \theta(p_{n-1}, p_m)$ (102)
$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_m)\}$ (103)
$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{n+1}, p_m)\}$ (104)

With (81) and (84), we obtain

$\theta(p_n, xp_m) \leq min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{n+1}, p_m)\}$ (105)
$\leq min\{l^{n-1}\theta(p_0, p_1), l^{n-2}\theta(p_0, p_1), \ldots, l^m\theta(p_0, p_1)\}$ (106)
$\leq min\{l^{n-1}, l^{n-2}, \ldots, l^m\}\theta(p_0, p_1)$ (107)

Taking the limit of $\theta(p_n, p_m)$ as $n, m \to \infty$, we have

$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e$. (108)

So, $\{p_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is convergent to s.

Suppose $Ts \neq s$

$\theta(p_n, Ts) \leq c[\theta(p_{n-1}, p_n) + \theta(s, Ts)]$. (109)

Taking the limit as $n \to \infty$ and using the fact that the function is continuous

in its variables, we get

$\theta(s, Ts) \leq c(\theta(s, Ts))$. (110)

Hence,

$\theta(s, Ts) \leq e$. (111)

This is a contradiction. So, $Ts = s$.

To show the uniqueness, suppose $w \neq s$ is such that $Tw = w$ and $Ts = s$, then

$\theta(Ts, Tw) \leq c(\theta(s, w))$. (112)

Since $Ts = s$ and $Tw = w$,

We have
$\theta(s, w) \leq e$. (113)

This implies that $w = s$.

Theorem 4.6: Let $(K, \theta, \ominus)$ be a complete operational metric space with an operation defined by $p \ominus q = min\{p, q\}$. Suppose $T : K \to K$ is a self map and there exists a real number c, satisfying $0 \leq c < 0.5$ for each $p, q \in K$ with

$\theta(Tp, Tq) \leq c[\theta(p, Tq) + \theta(q, Tp)]$ (114)

Then T has a unique fixed point.

Proof: Considering (94) with an arbitrary point $p_0 \in K$ and define a sequence p_0

by $p_n = T^n p_0$,

$\theta(p_n, p_{n+1}) = \theta(Tp_{n-1}, Tp_n) \leq c[\theta(p_{n-1}, p_{n+1}) + \theta(p_n, p_n)]$. (115)

(95) implies

$\theta(p_n, p_{n+1}) \leq \frac{c}{1-c} \theta(p_{n-1}, p_n)$ (116)

If $= \frac{c}{1-c}$, then.

$\theta(p_n, p_{n+1}) \leq l\theta(p_{n-1}, p_n)$. (117)

Suppose T satisfies condition (97), then

$\theta(p_n, p_{n+1}) \leq l(\theta(p_{n-1}, p_n))$ (118)

$\leq l^2(\theta(p_{n-2}, p_{n-1}))$ (119)

Using this repeatedly, we obtain

$\theta(p_n, p_{n+1}) \leq l^n(\theta(p_0, p_1))$. (120)

By using (θ_4) of Definition 1.1 with $n > m$, we have

$\theta(p_n, p_m) \leq \theta(p_n, p_{n-1}) \ominus \theta(p_{n-1}, p_m)$ (121)

$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_m)\}$ (122)

$= min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{m+1}, p_m)\}$ (123)

With (100) and (103), we obtain

$b(p_n, p_m) \leq min\{\theta(p_n, p_{n-1}), \theta(p_{n-1}, p_{n-2}), \ldots, \theta(p_{m+1}, p_m)\}$ (124)

$\leq min\{l^{n-1}\theta(p_0, p_1), l^{n-2}\theta(p_0, p_1), \ldots, l^m\theta(p_0, p_1)\}$ (125)

$\leq min\{l^{n-1}, l^{n-2}, \ldots, l^m\}\theta(p_0, p_1)$ (126)

Taking the limit of $\theta(p_n, p_m)$ as $n, m \to \infty$, we have

$\lim_{n,m\to\infty} \theta(p_n, p_m) \to e$. (127)

So, $\{p_n\}$ is a S-Cauchy Sequence.

By the completeness of $(K, \theta, \ominus)$, there exists $s \in K$ such that $\{p_n\}$ is convergent to s.

Suppose $Ts \neq s$

$\theta(p_n, Ts) \leq c[\theta(p_{n-1}, Ts) + \theta(s, p_n)]$. (128)

Taking the limit as $n \to \infty$ and using the fact that the function is continuous

in its variables, we get

$\theta(s, Ts) \leq c(\theta(s, Ts))$. (129)

Hence,

$\theta(s, Ts) \leq e$. (130)

This is a contradiction. So, $Ts = s$.

To show the uniqueness, suppose $w \neq s$ is such that $Tw = w$ and $Ts = s$,

Then

$\theta(Ts, Tw) \leq c(\theta(s, w))$. (131)

Since $Ts = s$ and $Tw = w$, we have

$\theta(s, w) \leq$ (132)

This implies that $w = s$.

RESULTS AND DISCUSSION

Main Theorem

Building on the lemmas established in Section 2, we proved a generalized Ćirić fixed point theorem within the framework of operational metric spaces. Specifically, we showed that if a self-map $Q: K \to K$ satisfies a generalized contractive inequality of the form:

$$Q(hp, hq) \leq c\, max\{Q(p, q), Q(p, hq), Q(q, hp), Q(p, hp), Q(q, hq)\}.$$

For all $p, q \in K$, then Q possesses a unique fixed point in K. The proof proceeds by constructing an iterative sequence $\{p_n\}$ defined by $p_{n+1} = Qp_n$ for some $p_0 \in K$, and showing that $\{p_n\}$ forms a Cauchy sequence under the operational metric θ. Completeness ensures convergence to a point $p^* \in K$, which is then shown to satisfy $Qp^* = p^*$.

Uniqueness of the Fixed Point

Uniqueness follows directly from the contractive condition. If p^* and q^* are both fixed points, applying the contractive inequality yields:

$$\theta(p *, q *) \leq \alpha\, \theta(p *, q *),$$

which implies $\theta(p *, q *) = 0\ and\ hence\ p * = q *$.

Special Cases

When the binary operation $\ominus$ is chosen as addition, maximum, or minimum, the theorem reduces to specific forms that generalize earlier fixed point results in the literature:

1. **Addition-based metrics** recover Banach-type theorems in extended metric spaces.
2. **Maximum-based metrics** connect to results in b-metric spaces and partial metric spaces.
3. **Minimum-based metrics** provide novel contraction scenarios not previously addressed.

These cases confirm the versatility of the operational metric framework in accommodating a broad class of contraction mappings.

Illustrative Examples

To validate the theoretical results, examples are constructed where K is a set equipped with a well-defined binary operation $\ominus$ and an operational metric θ satisfying completeness. For each case, a mapping T is explicitly defined to meet the generalised contractive condition, and the iterative process is shown to converge to the unique fixed point predicted by the theorem.

DISCUSSION

The results demonstrate that incorporating a binary operation into the metric structure extends the applicability of fixed point theory beyond classical settings. This operational perspective allows for greater flexibility in modeling problems from applied mathematics, where distance measures often interact with additional algebraic structures. The generalized Ćirić-type theorem presented here unifies and extends several existing results, offering a framework that can be adapted to specialized spaces and problem contexts.

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