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On Metric Quasigroups and Metric Loops

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Abstract

Metric space is a set in which a notion of distance between elements is defined in a precise and consistent way. Metric space provides a general and flexible framework for studying the idea of distance which underlies much of mathematics and its applications. In 1922, Stefan Banach was able to use the space alongside with contraction to prove the existence and uniqueness of solutions of ODE. This study opened the eyes of many researchers to this area of study. In this paper, we introduce the concept of metric quasigroups and metric loop with examples. This will enrich the space. We also introduce homomorphism in this concept. Our work unify, extend and generalize existing work in literature.

Keywords: Metric quasigroup, Metric group, Metric loop, Homomorphism

INTRODUCTION

Metric space is a set in which a notion of **distance** between elements is defined in a precise and consistent way. Metric space provides a **general and flexible framework** for studying the idea of *distance* which underlies much of mathematics and its applications. In 1906, **Maurice Fréchet** formally introduced the concept of a **metric space** in his doctoral thesis. A metric space is a set X with a distance function $d: X \times X \to \mathbb{R}^+$ satisfying Positivity, Symmetry and Triangle inequality Frechet (1906). Sixteen years later, Banach contraction principle (BCP) was introduced by Stefan Banach (see Banach (1922)). This was used as alternative method to prove the existence and uniqueness of solutions of ordinary differential equation. Seeing this, many authors picked interest in this area of study. Some of them tried to generalize metric space while some either work on contraction or iterative method used (see Adewale et al. (2024), Adewale et al. (2025), Ayodele et al. (2024), Ayodele et al. (2025), Loyinmi et al. (2024) and others in the references). In 2025, Adewale et al. introduced metric space with binary operation. This study introduced metric space with algebraic operations like addition, multiplication and any other operations. This concept leads to combined structures like metric groups and topological groups. The concept of metric space with binary operation introduced by Adewale et al. (2025) was later generalized to quasi metric space with binary operation by Ayodele et al. in the same year.

PRELIMINARIES

Definition 2.1 Adewale et al. (2025): Let K be a non-empty set, ∇, a binary operation with e as its identity element and $\rho : K \times K \to \mathbb{R}^+$. ρ is called an operational metric or metric with binary operation if the following axioms are satisfied:

$\rho_1: \rho(m,n) \geq e$;
$\rho_2: \rho(m,n) = e$ if and only if $m = n$;
$\rho_3 : \rho(m,n) = \rho(n,m)$;
$\rho_4 : \rho(m,n) \leq \rho(m,r) \nabla b(r,n)$ $for\ all\ m,n,r \in K.$

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K together with ρ is called an operational metric space or metric space with binary operation. It is denoted by (K, ρ, ∇).

Remark 2.2:
i. If the binary operation ∇ is defined by $p \nabla q = p + q$, the Definition 2.1 reduces to metric space introduced by Fretchet (1906) .
ii. If the binary operation ∇ is defined by $p \nabla q = p \times q$, the Definition 2.1 reduces to multiplicative metric space introduced by Bashirov et al (2008).

Definition 2.3 Ayodele et al. (2025): Let K be a non-empty set, ∇, a binary operation with e as its identity element and $\rho : K \times K \to \mathbb{R}^+$. ρ is called an operational metric or metric with binary operation if the following axioms are satisfied:
$\rho_1 : \rho(m,n) \geq e$;
$\rho_2 : \rho(m,n) = e$ if and only if $m = n$;
$\rho_3 : \rho(m,n) \leq \rho(m,r) \nabla b(r,n)\ for\ all\ m,n,r \in K.$
K together with ρ is called an operational quasi metric space or quasi metric space with binary operation. It is denoted by (K, ρ, ∇).

Remark 2.4:
i. If the binary operation ∇ is defined by $p \nabla q = p + q$, the Definition 2.3 reduces to quasi metric space introduced by Wilson (1931).
ii. If the binary operation ∇ is defined by $p \nabla q = p \times q$, the Definition 2.1 reduces to multiplicative quasi metric space.

RESULTS

Metric Quasigroups and Metric Loop

Definition 3.1: A quasigroup is a set Q with a binary operation $\oslash$ such that:
For every $a, b \in Q$, there exist unique elements $x, y \in Q$, satisfying:

i. $a \oslash x = b$
ii. $y \oslash a = b$

$(Q, \oslash)$ is a quasigroup because every row and every column contains each element exactly once and you can uniquely solve $a \oslash x = b$ and $y \oslash a = b$. If d is a metric on Q, then $(Q, d, \oslash)$ is a metric quasigroup.

Example 3.2: Let $Q = \mathbb{R}$ be a set of real numbers. If $\oslash$ and $d: X \times X \to \mathbb{R}^+$ are defined by $x \oslash y = x - y$ and $d(x,y) = |y - x|$, then $(Q, d, \oslash)$ is a metric quasigroup.

Remark 3.3

i. You can always solve equations of the form $a \oslash x = b$ and $y \oslash a = b$.
ii. Solutions are unique.
iii. A metric quasigroup does not require an identity element or associativity.
iv. The operation is isometric in each variable, that is, for all $a, b, c \in Q$, $d(a \oslash c,\ b \oslash c) = d(a,b)$ and $d(c \oslash a,\ c \oslash b) = d(a,b)$.

Verification

(i) $d(a \oslash c, b \oslash c) = |(a - c) - (b - c)|$
$= |a - c - b + c|$
$= |a - b|$
$= d(a,b)$

(ii) $d(c \oslash a, c \oslash b) = |(c - a) - (c - b)|$
$= |c - a - c + b|$
$= |-a + b|$
$= |b - a|$
$= |a - b| = d(a,b)$

Example 3.4: Consider the set $Q = \{1,2,3\}$ with the operation given by the table:

$\oslash$	1	2	3
1	2	3	1
2	3	1	2
3	1	2	3

If $d: Q \times Q \to \mathbb{R}^+$ is defined by $d(x,y) = |y - x|$, then $(Q, d, \oslash)$ is a metric quasigroup.

Example 3.5: Consider the set $Q = \mathbb{Z}_6 - \{0\}$ with the operation given by this table:

$\oslash$	1	2	3	4	5
1	2	4	5	1	3
2	1	3	2	4	5
3	3	2	4	5	1
4	5	1	3	2	4
5	4	5	1	3	2

If $d: Q \times Q \to \mathbb{R}^+$ is defined by $d(x,y) = |y - x|$, then $(Q, d, \oslash)$ is a metric quasigroup.

Definition 3.6: A loop is a quasigroup with an identity element**.** Let $(X, \sqcap)$ be a loop and d, a metric on X, $(X, d, \sqcap)$ is a metric loop.

Example 3.7: Consider the set $X = \{1,2,3,4\}$ with the operation given by this table:

$\sqcap$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

If $d: X \times X \to \mathbb{R}^+$ is defined by $d(x, y) = |y - x|$, then $(X, d, \sqcap)$ is an associative metric loop.

Example 3.8: Consider the set $X = \{1,2,3,4,5\}$ with the operation given by this table:

$\sqcap$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	1	2	4
4	4	3	5	1	2
5	5	4	2	3	1

If $d: X \times X \to \mathbb{R}^+$ is defined by $d(x, y) = |y - x|$, then $(X, d, \sqcap)$ is an non associative metric loop. It is non associative because $3\sqcap(4\sqcap5) = 3\sqcap2 = 5$ and $(3\sqcap4)\sqcap5 = 2\sqcap5 = 3$.

HOMOMORPHISM

Definition 3.9: Let $(G,*, d_G)$ and $(F,*, d_F)$ be two **metric groups** equipped with a metric d such that the groups' operations are compatible with the metric. A map $g: G \to F$ is said to be metric group homomorphism if $g(x * y) = g(x) * g(y)$for all $x, y \in G$.

Remark 3.10
A **homomorphism in metric groups** is a group homomorphism $g: G \to F$ considered together with its metric properties.

a) Continuous homomorphism: g is continuous with respect to d_G and d_F . In metric groups, continuity at the identity implies continuity everywhere.
b) Lipschitz homomorphism: $d_F(g(x), g(y)) \leq k\, d_G(x, y)$for all $x, y \in G$.
c) Isometric homomorphism: $d_F(g(x), g(y)) = d_G(x, y)$. This is an injective homomorphism preserving distances.

Example 3.11: Let $(G, +, d_G)$ and $(F, +, d_F)$ be two metric groups. If $g: G \to F$ is a map and $g(x) = cx$**,** then $g: (G, +, d_G) \to (F, +, d_F)$ is a metric group homomorphism with a usual metric.

Definition 3.12: Let $(A,*, d_A)$ and $(B,\ominus, d_B)$ be two metric quasigroups equipped with a metric d such that the operations are compatible with the metric. A map $h: A \to B$ is said to be metric quasigroup homomorphism if $h(x * y) = h(x) \ominus h(y)$for all $x, y \in A$.

This automatically respects left and right division:

$h(x\backslash y) = h(x)\backslash h(y)$, $h(y/x) = h(y)/h(x)$.

Definition 3.13: Let $(A,\boxdot, d_A)$ and $(B, \oplus, d_B)$ be two **metric loops** equipped with a metric d such that the operations are compatible with the metric. A map $\mu: A \to B$ is said to be metric loop homomorphism if $\mu(x \boxdot y) = \mu(x)\oplus\mu(y)$for all $x, y \in A$.

Remark 3.14
a) It preserves identity: $\mu(e_A) = e_B$.
$\mu(e_A \boxdot y) = \mu(y)$
$\mu(e_A)\oplus\mu(y) = e_B\oplus\mu(y) = \mu(y)$
b) If the metric is **translation-invariant** (left or right invariant) then μ is continuous everywhere if and only if μ is continuous at the identity.

Example 3.15: A normed vector space under addition is a loop. Linear maps are continuous homomorphism

Example 3.16: Let $Q = \mathbb{R} - \{0\}$ with operation

$$x \circledast y = \frac{x}{y}.$$

With the usual metric, multiplication and division are continuous. The map $\mu(x) = | x |$ is **not** a homomorphism; $\mu(x) = x^2$ *is* a homomorphism and continuous.

MONOMORPHISM

Definition 3.17: Let $(G,*,d_G)$ and $(F,*,d_F)$ be two metric groups equipped with a metric d such that the groups' operations are compatible with the metric. A map $g: G \to F$ is said to be metric group monomorphism if:

a. $g(x * y) = g(x) * g(y)$for all $x, y \in G$.
b. g is injective.

Definition 3.18: Let $(A,*,d_A)$ and $(B,\ominus,d_B)$ be two metric quasigroups equipped with a metric d such that the operations are compatible with the metric. A map $h: A \to B$ is said to be metric quasigroup monomorphism if:

a. $h(x * y) = h(x) \ominus h(y)$for all $x, y \in A$.
b. h is injective.

Definition 3.19: Let $(A,\boxdot,d_A)$ and $(B,\oplus,d_B)$ be two metric loops equipped with a metric d such that the operations are compatible with the metric. A map $\mu: A \to B$ is said to be metric loop monomorphism if

a. $\mu(x \boxdot y) = \mu(x)\oplus\mu(y)$for all $x, y \in A$.
b. μ is injective.

EPIMORPHISM

Definition 3.20: Let $(G,*,d_G)$ and $(F,*,d_F)$ be two metric groups equipped with a metric d such that the groups' operations are compatible with the metric. A map $g: G \to F$ is said to be metric group epimorphism if:

c. $g(x * y) = g(x) * g(y)$for all $x, y \in G$.
d. g is surjective.

Definition 3.21: Let $(A,*,d_A)$ and $(B,\ominus,d_B)$ be two metric quasigroups equipped with a metric d such that the operations are compatible with the metric. A map $h: A \to B$ is said to be metric quasigroup epimorphism if:

c. $h(x * y) = h(x) \ominus h(y)$for all $x, y \in A$.
d. h is surjective.

Definition 3.22: Let $(A,\boxdot,d_A)$ and $(B,\oplus,d_B)$ be two metric loops equipped with a metric d such that the operations are compatible with the metric. A map $\mu: A \to B$ is said to be metric loop epimorphism if

c. $\mu(x \boxdot y) = \mu(x)\oplus\mu(y)$for all $x, y \in A$.
d. μ is surjective.

ISOMORPHISM

Definition 3.23: Let $(G,*,d_G)$ and $(F,*,d_F)$ be two metric groups equipped with a metric d such that the groups' operations are compatible with the metric. A map $g: G \to F$ is said to be metric group isomorphism if:

e. $g(x * y) = g(x) * g(y)$for all $x, y \in G$.
f. g is bijective.

Definition 3.24: Let $(A,*,d_A)$ and $(B,\ominus,d_B)$ be two metric quasigroups equipped with a metric d such that the operations are compatible with the metric. A map $h: A \to B$ is said to be metric quasigroup isomorphism if:

e. $h(x * y) = h(x) \ominus h(y)$for all $x, y \in A$.
f. h is bijective.

Definition 3.25: Let $(A,\boxdot,d_A)$ and $(B,\oplus,d_B)$ be two metric loops equipped with a metric d such that the operations are compatible with the metric. A map $\mu: A \to B$ is said to be metric loop isomorphism if

e. $\mu(x \boxdot y) = \mu(x)\oplus\mu(y)$for all $x, y \in A$.
f. μ is bijective.

ENDOMORPHISM

Definition 3.26: Let $(G,*,d_G)$ be a metric group equipped with a metric d such that the operation is compatible with the metric. A map $g: G \to G$ is said to be metric group endomorphism if $g(x * y) = g(x) * g(y)$for all $x, y \in G$.

Definition 3.27: Let $(A,\ominus,d_A)$ be a metric quasigroup equipped with a metric d such that the operation is compatible with the metric. A map $h: A \to A$ is said to be metric quasigroup endomorphism if $h(x \ominus y) = h(x) \ominus h(y)$for all $x, y \in A$.

Definition 3.28: Let $(A,\boxdot,d_A)$ be a metric loop equipped with a metric d such that the operations is compatible with the metric. A map $\mu: A \to A$ is said to be metric loop endomorphism if $\mu(x \boxdot y) = \mu(x) \boxdot \mu(y)$for all $x, y \in A$.

AUTOMORPHISM

Definition 3.29: Let $(G,*, d_G)$ be a metric groups equipped with a metric d such that the group's operations is compatible with the metric. A map $g: G \to G$ is said to be metric group automorphism if:

g. $g(x * y) = g(x) * g(y)$for all $x, y \in G$.
h. g is bijective.

Definition 3.30: Let $(A,*, d_A)$ be a metric quasigroups equipped with a metric d such that the operation is compatible with the metric. A map $h: A \to A$ is said to be metric quasigroup automorphism if:

g. $h(x * y) = h(x) * h(y)$for all $x, y \in A$.
h. h is bijective.

Definition 3.31: Let $(A,\boxdot, d_A)$ be a metric loop equipped with a metric d such that the operation is compatible with the metric. A map $\mu: A \to A$ is said to be metric loop automorphism if

g. $\mu(x \boxdot y) = \mu(x) \boxdot \mu(y)$for all $x, y \in A$.

μ is objective

CONCLUSION

In this study, we introduce the concept of metric quasigroups and metric loop with examples. This will enrich the space. We also introduce homomorphism in this concept.

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